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An integrable Hamiltonian motion on a sphere: II. The separation of variables

B Gaffet

CEA DSM/DAPNIA/Service d'Astrophysique, CEN Saclay, 91191-Gif sur Yvette Cedex, France

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Abstract. We consider the motions on the unit sphere $x^2 + y^2 + z^2 = 1$, in a potential $V \propto 1/(xyz)^{2/3}$, which have been shown earlier to derive from a *completely integrable* Hamiltonian. The separation of variables is completed in terms of an independent variable u which differs from the canonical time t: $3u = 9(mt + I_2\Phi) - S$ where m and I_2 are the two integrals of motion, S is the action, and Φ is the integration constant (constant along each trajectory). Finally a Lax pair is deduced from a separable form of the system (Lax P D 1968 *Commun. Pure Appl. Math.* **21** 467).

1. Introduction

It has recently been noted (Gaffet 1996) that the Hamiltonian in three-dimensional flat space, defined by the equations of motion

$$xx''(t) = yy''(t) = zz''(t) = \frac{\text{constant}}{(xyz)^{2/3}}$$

was reducible, as a consequence of the separability of the radial motion, to another Hamiltonian governing the two-dimensional motion of a particle on a unit sphere. In the same work, it was realized that the equations governing these spherical motions possessed the Painlevé property (Ince 1956), and a second integral of the motion was found to be present, in agreement with a well known conjecture (Ablowitz and Segur 1977). The Hamiltonian considered is thus *completely integrable*, and presumably endowed with the usually associated remarkable properties, such as the existence of a Bäcklund transformation (BT) connecting the different particular solutions. A direct search for the BT seemed difficult however, and we chose to look instead for a formulation in terms of the separated variables: if such a formation could be obtained, a BT ought to be easily deduced.

In Gaffet (1998) (hereafter referred to as paper I) a certain form of separation of variables was achieved, assuming a special relation ($m = m_0(I_2)$) between the two integrals of the motion m and I_2, which ensured that the quadratures involved were elliptic. In the present work, the results of paper I are generalized to fully arbitrary values of the two integrals of the motion, and the general solution is found to be represented by hyper-elliptic integrals.

2. General description of the system

2.1. The surface (S) and its parametrization

The equations of motion considered here are those governing point-mass motion on the unit sphere $x^2 + y^2 + z^2 = 1$, in a potential V_s

$$V_s = \frac{3/2}{(xyz)^{2/3}} \tag{2.1}$$

they have the form (paper I, equation 1.7)

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{U'}{V\sqrt{U}}\right) = \frac{2}{3}\left(\frac{1-U^3}{U^{3/2}}\right) \tag{2.2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{V'}{U\sqrt{V}}\right) = \frac{2}{3}\left(\frac{1-V^3}{V^{3/2}}\right) \tag{2.2b}$$

in coordinates $U \equiv (y/x)^{2/3}$ and $V \equiv (z/x)^{2/3}$. The independent variable u, which does not coincide with the canonical time t, is the following integral over time

$$u = \int \frac{\mathrm{d}t}{(xyz)^{2/3}} \tag{2.3}$$

and the prime in U' and V' denotes derivation with respect to u.

The velocity variables are conveniently represented in the form of a three-vector $\boldsymbol{\xi}$ with Cartesian components ξ, η, ζ

$$\begin{aligned} h\xi &\equiv (VU' - UV') \\ h\eta &\equiv \frac{V'}{U^2} \\ h\zeta &\equiv -\frac{U'}{V^2} \end{aligned} \tag{2.4}$$

subject to the constraint

$$(\xi + \eta U^3 + \zeta V^3) = 0 \tag{2.5}$$

and h is a constant, taken to be $h = I_2/2$ in paper I; but in the present paper we choose

$$h = \frac{2}{\sqrt{3}}$$

instead, in order not to exclude the cases of the vanishing I_2 from consideration.

The two integrals of the motion, m and I_2, are, respectively, represented by the following equations

$$(U^3 + V^3 + 1 - 3mUV) + (\xi^2 + \eta^2 U^3 + \zeta^2 V^3) = 0 \tag{2.6}$$

$$(\xi + \eta + \zeta) = \frac{I_2}{h} + \xi\eta\zeta \tag{2.7}$$

which, together with equation (2.5), implicitly determine the velocity vector $\boldsymbol{\xi}$ in terms of the coordinates U and V. In paper I it was suggested to replace this system (2.5)–(2.7) by a simpler system involving three variables only: π, ρ, η, through the elimination of the remaining variables U, V, ξ and ζ, in the following way. Defining

$$\pi \equiv \frac{mV}{U^2} \tag{2.8}$$

$$\rho \equiv \pi(\xi\zeta - 1) \tag{2.9}$$

the variables ξ and ζ are the two roots of the second-degree equation

$$\pi\xi^2 - \xi(\eta\rho + 2\pi\sqrt{\varepsilon}) + (\rho + \pi) = 0 \qquad \left(\sqrt{\varepsilon} = \frac{I_2}{2h} = \frac{I_2\sqrt{3}}{4}\right) \tag{2.10}$$

which has the discriminant Δ

$$\Delta \equiv (\eta\rho + 2\pi\sqrt{\varepsilon})^2 - 4\pi(\rho + \pi) \tag{2.11}$$

and is solved as

$$2\pi\xi = (\eta\rho + 2\pi\sqrt{\varepsilon}) + \sqrt{\Delta} \qquad 2\pi\zeta = (\eta\rho + 2\pi\sqrt{\varepsilon}) - \sqrt{\Delta}. \tag{2.12}$$

In the same way, U^3 is the root of the second-degree equation (see equation 2.5)

$$\zeta\frac{\pi^3}{m^3}U^6 + \eta U^3 + \xi = 0 \tag{2.13}$$

which has the discriminant D

$$D \equiv \eta^2 - \frac{4\pi^2}{m^3}(\rho + \pi) \tag{2.14}$$

and is solved as

$$U^3 = \frac{m^3(\sqrt{D} - \eta)}{2\zeta\pi^3}. \tag{2.15}$$

The algebraic system (2.5)–(2.7) defining $\boldsymbol{\xi}(U, V)$ is thereby reduced to a single algebraic equation, representing a surface (S) by the coordinates (η, π, ρ); although the equation is complicated, it can be written rather compactly in the following way

$$\eta(\eta\rho + \pi\varphi'(\eta)) + 2\pi R(\rho + \pi) + \sqrt{D\Delta} = 0 \tag{2.16}$$

where

$$\varphi(\eta) \equiv (\eta^2 - 2\eta\sqrt{\varepsilon} + 1) \qquad \varphi'(\eta) \equiv \frac{\mathrm{d}\varphi}{\mathrm{d}\eta} \tag{2.17}$$

and

$$R \equiv \frac{[3\pi - \varphi(\eta)]}{\rho}. \tag{2.18}$$

Achieving the separation of variables essentially amounts to finding a simple parametric representation of that surface; although what is meant by 'simple' is not completely clear. In the light of the results of paper I and of the present work, it seems that, either the parametric representation ought to be fully rational (as was the case in paper I); or, if it must involve a radical, the polynomial in two variables under the radical ought to be fully decomposable (into linear factors).

A general property of the surface (S) in the present problem, revealed in paper I, is that it presents *a line of singularity* (the locus of double points of the plane sections) at

$$\rho = 0 \qquad 3\pi = \varphi(\eta). \tag{2.19}$$

As usual, choosing as new variable the slope R at the double point (see equation 2.18) will eliminate the singularity. Choosing then R and η as new independent variables, *the equation of the surface (S) reduces to a mere second-degree equation for* $\rho(R, \eta)$

$$A\rho^2 + B\rho + C = 0 \tag{2.20}$$

where

$$A(R,\eta) \equiv 4R^2(\varepsilon-1) + 12R(\eta\sqrt{\varepsilon}-1) + 9\eta^2$$
$$B(R,\eta) \equiv 3m^3R^2(R+3) + 4\varphi(\eta)[2R(\varepsilon-1) + 3(\eta\sqrt{\varepsilon}-1)] \tag{2.21}$$
$$C(R,\eta) \equiv 3m^3R^2\varphi + 18m^3R(\eta^2-\eta\sqrt{\varepsilon}) + 27m^3\eta^2 + 4\varphi^2(\varepsilon-1).$$

Remarkably, its discriminant, B^2-4AC, which is a sixth-degree polynomial in R and η, turns out to be *fully decomposable into a product of six linear factors*, as will be shown later.

2.2. The differential system in the new coordinates R and η

We now transform our original differential system (2.4)–(2.7) to the new coordinate system R, η and ρ; that is, we calculate the derivatives $R'(u)$, $\eta'(u)$ and $\rho'(u)$.

First, the derivative of $\pi \equiv mV/U^2$ is, quite simply

$$\pi'(u) = hm\sqrt{D}. \tag{2.22}$$

Then, we remark that, since $V'/U\sqrt{V} \equiv h\eta\sqrt{m/\pi}$, the equation of motion (2.2*b*) directly produces the derivative of η^2/π; it reads

$$(2\pi\eta' - \eta\pi') = hm\frac{(1-V^3)}{U^3}. \tag{2.23}$$

Substituting

$$\frac{V^3}{U^3} \equiv \frac{(\sqrt{D}-\eta)}{2\zeta} \tag{2.24a}$$

$$\frac{1}{U^3} \equiv \frac{-\zeta\pi(\sqrt{D}+\eta)}{2(\rho+\pi)} \tag{2.24b}$$

we obtain

$$\frac{(V^3-1)}{U^3} \equiv \frac{\pi}{2(\rho+\pi)}[\xi(\sqrt{D}-\eta) + \zeta(\sqrt{D}+\eta)] \equiv \frac{[(\eta\rho+2\pi\sqrt{\varepsilon})\sqrt{D} - \eta\sqrt{\Delta}]}{2(\rho+\pi)} \tag{2.25}$$

π' having been already computed, equation (2.23) thus yields η', in the form

$$\eta'(u) = \frac{hm}{4\pi(\rho+\pi)}[\eta(\sqrt{\Delta}+\rho\sqrt{D}) + \pi\varphi'(\eta)\sqrt{D}]. \tag{2.26}$$

The same equations (2.24*a*, *b*) may also be used to determine an alternative expression for R, equivalent to (2.18), in the following way

$$\frac{(V^3+1)}{U^3} \equiv \frac{\pi}{2(\rho+\pi)}[\xi(\sqrt{D}-\eta) - \zeta(\sqrt{D}+\eta)] \equiv \frac{[\sqrt{D\Delta} - \eta(\eta\rho+2\pi\sqrt{\varepsilon})]}{2(\rho+\pi)}$$
$$\equiv -(\eta^2+\pi R) \tag{2.27}$$

to establish the last equality, use has been made of equation (2.16) of the surface (S). This new expression for R reads

$$-mR \equiv \frac{(V^3+1)}{UV} + \frac{m\eta^2}{\pi}. \tag{2.28}$$

The derivative $R'(u)$ may be found through differentiation of the above expression. The derivative of its first term is easily obtained

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{V^3+1}{UV}\right) = \frac{h\pi}{m}(\zeta-\xi) + h\eta\frac{U}{V^2}(V^3-1)$$

while that of the second term is directly given by (2.23)

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{m\eta^2}{\pi}\right) = -h\eta\frac{U}{V^2}(V^3 - 1).$$

There is a cancellation of terms, and we obtain the following simple result

$$R'(u) = \frac{h}{m^2}\sqrt{\Delta}. \tag{2.29}$$

The derivative of ρ is also of interest

$$\rho'(u) = \frac{hm}{2\pi}(\sqrt{\Delta} + \rho\sqrt{D}). \tag{2.30}$$

Since R' and π' are respectively proportional to $\sqrt{\Delta}$ and $\sqrt{D}$, this equation may also be written

$$(2\pi\rho' - \rho\pi') = m^3 R'. \tag{2.31}$$

In the same way, expression (2.26) of $\eta'(u)$ becomes

$$(\rho + \pi)\eta' = \frac{\eta}{2}\rho' + \frac{\varphi'(\eta)}{4}\pi' = \frac{1}{4\pi}[m^3\eta R' + \pi'(\eta\rho + \pi\varphi'(\eta))]. \tag{2.32}$$

Let us now form the combination

$$Z \equiv 2\eta\eta' + (R\pi' - 2\pi R')$$

upon substitution of expression (2.32) of η', and making use of equation (2.16) of the surface, we find

$$2\pi(\rho + \pi)Z = m^3 R' D - \pi'\sqrt{D\Delta} = 0.$$

Thus, η' admits the simpler formulation

$$2\eta\eta' = (2\pi R' - R\pi'). \tag{2.33}$$

Still another simple relation between derivatives may be deduced through differentiation of the definition (2.18) of R

$$\rho R' + R\rho' - 3\pi' + \varphi'(\eta)\eta' = 0. \tag{2.34}$$

Substitution of expression (2.31) and (2.33) of ρ' and η' yields a relation homogeneous in R' and π'

$$\frac{R'(u)}{\pi'(u)} \equiv \frac{\sqrt{\Delta}}{m^3\sqrt{D}} = \frac{[3 + (R/2\pi)((\pi\varphi'(\eta)/\eta) - \rho)]}{[m^3(R/2\pi) + ((\pi\varphi'(\eta)/\eta) + \rho)]} \tag{2.35}$$

which may be viewed as an alternative, equivalent form of the equation of the surface.

To sum up, we have obtained four simple linear homogeneous relations (2.31)–(2.34) between the derivatives of η, R, ρ and π, which may be written more compactly as

$$M\begin{bmatrix}\eta'\\ \rho'\\ R'\\ \pi'\end{bmatrix} = 0. \tag{2.36}$$

It is quite remarkable that the (4×4) matrix M should be symmetric

$$M = \begin{bmatrix} -4(\rho+\pi) & 2\eta & 0 & \varphi'(\eta)\\ 2\eta & 0 & -2\pi & R\\ 0 & -2\pi & m^3 & \rho\\ \varphi'(\eta) & R & \rho & -3 \end{bmatrix}. \tag{2.37}$$

The equation of the surface (S) is, of course

$$\det M = 0. \tag{2.38}$$

Since, when η and R are given, π is a linear function of ρ (see equation 2.18), this equation appears to be cubic in ρ. However, the terms cubic homogeneous in ρ and π, which amount to

$$16\pi(\rho + \pi)(3\pi - \rho R)$$

are really *second-degree terms* in ρ, since, by equation (2.18), $(3\pi - \rho R) = \varphi(\eta)$. Expanding the (4×4) determinant yields, of course, the second-degree equation (2.20).

3. The separation of variables

3.1. Complete decomposability of the discriminant $B^2 - 4AC$

The discriminant of the equation (2.20) giving $\rho(\eta, R)$, reads

$$\begin{aligned}(B^2 - 4AC) \equiv{} & 9m^6R^4(R+3)^2 - 72m^3R^3[\eta^3\sqrt{\varepsilon} - 3\eta^2 + 3\eta\sqrt{\varepsilon} + (1-2\varepsilon)] \\ & -108m^3R^2[\eta^4 + 4\eta^3\sqrt{\varepsilon} - 9\eta^2 + 2\eta\sqrt{\varepsilon} + 2] - 648m^3R\eta^2[\eta^2 + \eta\sqrt{\varepsilon} - 2] \\ & +36[4\varphi^3(\eta) - 27m^3\eta^4].\end{aligned} \tag{3.1}$$

In particular, when R vanishes it is given by

$$B^2 - 4AC|_{R=0} \equiv 36A_6(\eta)$$

where

$$A_6(\eta) \equiv 4\varphi^3(\eta) - 27m^3\eta^4. \tag{3.2}$$

The six roots of $A_6(\eta)$ may be viewed as the solution of the system

$$3mk_i = \frac{\varphi(\eta_i)}{\eta_i} \tag{3.3a}$$

$$\eta_i = 4k_i^3. \tag{3.3b}$$

The six values of mk_i precisely coincide with the slopes of the asymptotic directions of the curve $(B^2 - 4AC) = 0$, which are

$$\eta \sim mk_iR$$

in addition, when R takes arbitrary values the roots of the discriminant become simply

$$\eta = \eta_i + mk_iR$$

meaning that $(B^2 - 4AC)$ is decomposable into the product of six linear factors of the form

$$(\eta - \eta_i) - mk_iR.$$

3.2. The new variables l_1, l_2, m_1, m_2

It is essential that these factors, through rescaling, can be given a form *quadratic in* η_i (see equation (3.3a))

$$R\varphi(\eta_i) + 3\eta_i(\eta_i - \eta). \tag{3.4}$$

Let us form the second degree equation

$$R\varphi(l) + 3l(l - \eta) = 0 \tag{3.5}$$

thereby defining two new variables l_1 and l_2, the roots of that equation; we also introduce the auxiliary variables S and P, m_1 and m_2

$$S \equiv (l_1 + l_2) = \frac{(3\eta + 2R\sqrt{\varepsilon})}{(R+3)} \tag{3.6a}$$

$$P \equiv l_1 l_2 = \frac{R}{(R+3)} \tag{3.6b}$$

$$m_1^2 \equiv 4\varphi_1^3 - 27m^3 l_1^4 \equiv A_6(l_1) \tag{3.7a}$$

$$m_2^2 \equiv 4\varphi_2^3 - 27m^3 l_2^4 \equiv A_6(l_2) \tag{3.7b}$$

where

$$\varphi_1 = \varphi(l_1) \qquad \varphi_2 \equiv \varphi(l_2).$$

The linear factors (3.4) that make up the discriminant, may now be written

$$(R+3)(l_1 - \eta_i)(l_2 - \eta_i) \tag{3.8}$$

and the discriminant itself is therefore proportional to the product $A_6(l_1)A_6(l_2) \equiv m_1^2 m_2^2$; more precisely, we obtain

$$B^2 - 4AC \equiv \frac{9m_1^2 m_2^2}{(P-1)^6}. \tag{3.9}$$

Transforming to the new coordinates l_1 and l_2—or, equivalently, S and P—we have

$$R \equiv \frac{3P}{(1-P)} \tag{3.10a}$$

$$\eta \equiv \frac{(S - 2P\sqrt{\varepsilon})}{(1-P)} \tag{3.10b}$$

$$\varphi(\eta) \equiv \frac{\varphi_1 \varphi_2}{(P-1)^2} \tag{3.11}$$

and the coefficients of equation (2.20) become

$$A \equiv \frac{9(l_1 - l_2)^2}{(P-1)^2} \tag{3.12a}$$

$$B \equiv \frac{3}{(P-1)^3}[4\varphi_1\varphi_2(P + 1 - S\sqrt{\varepsilon}) - 27m^3 P^2]. \tag{3.12b}$$

The solution of equation (2.20) is accordingly

$$\rho \equiv \left[\frac{3m_1 m_2}{(P-1)^3} - B\right] \Big/ 2A \equiv \frac{[4\varphi_1\varphi_2(P + 1 - S\sqrt{\varepsilon}) - 27m^3 P^2 - m_1 m_2]}{6(l_1 - l_2)^2(1-P)}. \tag{3.13}$$

It is worth noting that both ρ and π (see equation 2.18) are functions linear in $m_1 m_2$, with coefficients rational in l_1 and l_2.

3.3. The differential system in separable form

We may now calculate the discriminants Δ and D, which are defined by equations (2.11) and (2.14)

$$\sqrt{\Delta} \equiv \frac{\sqrt{-m^3}}{h} \frac{(l_2^2 m_1 - l_1^2 m_2)}{(P-1)^2(l_2 - l_1)}. \tag{3.14}$$

The expression for $\sqrt{D}$ is more complicated, but can still be written compactly in the following way

$$\sqrt{D} \equiv \frac{2[3\pi(l_2^2 m_1 - l_1^2 m_2) - \eta(\varphi_2 l_2 m_1 - \varphi_1 l_1 m_2)]}{9h\sqrt{-m^3}P(P-1)(l_2 - l_1)}. \tag{3.15}$$

Using these expressions the derivatives $R'(u)$ and $\eta'(u)$, which are given by equations (2.29), (2.33) and (2.22), may be deduced

$$R'(u) \equiv \frac{(l_2^2 m_1 - l_1^2 m_2)}{\sqrt{-m}(P-1)^2(l_2 - l_1)} \tag{3.16a}$$

$$\eta'(u) \equiv \frac{(\varphi_2 l_2 m_1 - \varphi_1 l_1 m_2)}{3\sqrt{-m}(P-1)^2(l_2 - l_1)}. \tag{3.16b}$$

Differentiation of expressions (3.10) for R and η, yields S' and P'

$$\begin{aligned} S'(u) &\equiv (P-1)\left[(S - 2\sqrt{\varepsilon})\frac{R'}{3} - \eta'\right] \\ P'(u) &\equiv (P-1)^2\frac{R'}{3} \end{aligned} \tag{3.17}$$

hence

$$\begin{aligned} S'(u) &\equiv \frac{1}{3\sqrt{-m}}\frac{(l_2 m_1 - l_1 m_2)}{(l_2 - l_1)} \\ P'(u) &\equiv \frac{1}{3\sqrt{-m}}\frac{(l_2^2 m_1 - l_1^2 m_2)}{(l_2 - l_1)}. \end{aligned} \tag{3.18}$$

Finally, the separable form of the differential system is found

$$kl_1'(u) = \frac{-l_2 m_1}{(l_1 - l_2)} \qquad kl_2'(u) = \frac{+l_1 m_2}{(l_1 - l_2)} \tag{3.19}$$

where

$$k = 3\sqrt{-m}. \tag{3.20}$$

Its general solution is given by two quadratures

$$\int \frac{l_1\,\mathrm{d}l_1}{m_1} + \int \frac{l_2\,\mathrm{d}l_2}{m_2} = \Phi \tag{3.21}$$

where Φ is the integration constant; and the value of the independent variable u may be retrieved from the relation

$$\int \frac{\mathrm{d}l_1}{m_1} + \int \frac{\mathrm{d}l_2}{m_2} = \frac{u}{k}. \tag{3.22}$$

3.4. The canonical time

A general formula giving the canonical time t in differential form has already been given in paper I (equation 2.24)

$$\mathrm{d}t = \frac{UV}{\delta}\left[\mathrm{d}u + \frac{9}{8}XYZ\,\mathrm{d}\Phi\right]$$

where $\delta = (1 + U^3 + V^3)$ and $(9/8)XYZ \equiv (-3/2)h\xi\eta\zeta$; in terms of the new variables we have (see equations (2.27) and (2.9))

$$\frac{\delta}{U^3} \equiv 1 + \frac{(V^3+1)}{U^3} \equiv (1 - \eta^2) - \pi R \tag{3.23}$$

hence

$$\frac{\delta}{UV} \equiv -\frac{m}{\pi}[\pi R + \eta^2 - 1] \tag{3.24}$$

and

$$\xi\eta\zeta \equiv \frac{\eta(\rho+\pi)}{\pi}. \tag{3.25}$$

Thus, the differential dt involves the denominator $(\pi R + \eta^2 - 1)$, which is irrational since π is linear in $m_1 m_2$. The partial derivative $\partial t/\partial l_2$ has the general form

$$m_2 \frac{\partial t}{\partial l_2} = \frac{(n_0 + n_1 m_1 m_2)}{(d_0 + d_1 m_1 m_2)}$$

where n_0, n_1, d_0 and d_1 are rational functions of l_1 and l_2; and it can be reduced to the form

$$\frac{\partial t}{\partial l_2} = \frac{(N_0 + N_1 m_1 m_2)}{m_2 D_0} \tag{3.26}$$

where $D_0 \equiv (d_0 + d_1 m_1 m_2)(d_0 - d_1 m_1 m_2)$ is rational.

The rational part of the integral, $\int (N_1/D_0)\mathrm{d}l_2$, turns out to be calculable in closed form as a result of the decomposability of D_0, as we now show.

In the formula (3.23) giving δ, π is the linear function of ρ defined by equation (2.18)

$$3\pi \equiv \rho R + \varphi(\eta) \tag{3.27}$$

and ρ is expressed by (3.13); although complicated, the resulting expression of δ can be written compactly in the following way

$$\frac{\delta}{U^3} \equiv (1-P) + \frac{(\Sigma - 2P^2 m_1 m_2)}{4(P-1)^3(l_1-l_2)^2} \tag{3.28}$$

where

$$\Sigma = (m_1^2 l_2^4 + m_2^2 l_1^4). \tag{3.29}$$

Denoting δ^*, U^* as the new values of δ and U when $m_1 m_2$ changes sign, the product $\delta\delta^*/(UU^*)^3$ (which is essentially the denominator D_0) is found as

$$\frac{\delta\delta^*}{(UU^*)^3} \equiv (P-1)^2 + \frac{(\Sigma^2 - 4P^4 m_1^2 m_2^2)}{16(P-1)^6(l_1-l_2)^4} + \frac{\tilde{\Sigma}}{2(P-1)^2(l_1-l_2)^2} - \frac{(\Sigma + \tilde{\Sigma})}{2(P-1)^2(l_1-l_2)^2} \tag{3.30}$$

where

$$\tilde{\Sigma} \equiv (m_1^2 l_2^4 - m_2^2 l_1^4). \tag{3.31}$$

Noting that: $(\Sigma^2 - 4P^4 m_1^2 m_2^2) \equiv \tilde{\Sigma}^2$, this may be rewritten

$$\frac{\delta\delta^*}{(UU^*)^3} \equiv \left[(P-1) + \frac{\tilde{\Sigma}}{4(P-1)^3(l_1-l_2)^2}\right]^2 - \frac{m_1^2 l_2^4}{(P-1)^2(l_1-l_2)^2} \tag{3.32}$$

which is manifestly rationally decomposable into a product, as a function of l_2. As a result, the integral $\int (N_1/D_0)\,\mathrm{d}l_2$ is explicitly obtained

$$m_1 \int \frac{N_1}{D_0}\,\mathrm{d}l_2 = a_0 \omega_1$$

where a_0 is a constant when m and I_2 are fixed, and

$$\tanh\omega_1 = \frac{B_1}{m_1 l_2^2} \tag{3.33}$$

$$B_1 \equiv B(l_1, l_2) \equiv (P-1)^2(l_1 - l_2) + \frac{\tilde{\Sigma}}{4(P-1)^2(l_1-l_2)}. \tag{3.34}$$

The hyper-elliptic part of the integral, $\int(N_0\,\mathrm{d}l_2/m_2 D_0)$, must, by symmetry, contain a term ω_2

$$\tanh\omega_2 = \frac{B_2}{m_2 l_1^2} \tag{3.35}$$

where $B_2 \equiv B(l_2, l_1)$. When that part is subtracted out, what remains is a hyper-elliptic integral which no longer involves the denominator D_0, namely

$$\int \frac{\varphi_2\,\mathrm{d}l_2}{m_2}.$$

The canonical time is thus

$$\begin{cases} t = a_0(\omega_1 + \omega_2) + a_1\tau \\ \tau = \displaystyle\int \frac{\varphi_1\,\mathrm{d}l_1}{m_1} + \int \frac{\varphi_2\,\mathrm{d}l_2}{m_2} \end{cases} \tag{3.36}$$

where a_1 is a constant when m and I_2 are fixed. We remark that, unlike u and Φ, the time t does not satisfy the simple equation $\partial^2/\partial l_1\partial l_2 = 0$; rather, it is the related quantity τ that does.

4. The Bäcklund transformation

We consider in this section separable systems of the form

$$l_1'(u) = \frac{m_1}{(l_1 - l_2)} \qquad l_2'(u) = \frac{-m_2}{(l_1 - l_2)} \tag{4.1}$$

where m_i^2 $(i = 1, 2)$ is a sixth-degree polynomial in l_i, denotes $\mu^2(\lambda)$

$$\mu^2 \equiv (a_6\lambda^6 + \cdots + a_0). \tag{4.2}$$

These systems are integrable by quadratures

$$\begin{cases} \Phi = \displaystyle\int \frac{\mathrm{d}l_1}{m_1} + \int \frac{\mathrm{d}l_2}{m_2} \\ u = \displaystyle\int \frac{l_1\,\mathrm{d}l_1}{m_1} + \int \frac{l_2\,\mathrm{d}l_2}{m_2} \end{cases} \tag{4.3}$$

where Φ is the integration constant. The systems studied in the preceding sections are amenable to this form, by mere inversion of the variables l_i (see equation (3.21) and (3.22)), and the corresponding new form of $\mu^2(\lambda)$ (see equations (3.7) and (2.17)) is:

$$\mu^2(\lambda) \equiv 4[\lambda^2 - 2\lambda\sqrt{\varepsilon} + 1]^3 - 27m^3\lambda^2. \tag{4.4}$$

4.1. The Bäcklund transformation as a linear combination of generators

Let us introduce an auxiliary second-order differential system, whose integral curves generalize the curves l_i = constant, along which we have: $\mathrm{d}\Phi = \mathrm{d}\lambda/\mu$, $\mathrm{d}u = \lambda\,\mathrm{d}\lambda/\mu$, where λ may be identified with l_j $(j \neq i)$. These equations constitute a second-order differential system for the two unknown functions $l_i(\lambda)$ $(i = 1, 2)$

$$\frac{\mathrm{d}l_1}{m_1} + \frac{\mathrm{d}l_2}{m_2} = \frac{\mathrm{d}\lambda}{\mu} \qquad \frac{l_1\,\mathrm{d}l_1}{m_1} + \frac{l_2\,\mathrm{d}l_2}{m_2} = \frac{\lambda\,\mathrm{d}\lambda}{\mu} \tag{4.5}$$

which determines a two-parameter family of curves

$$l_1 = f(l_{10}, l_{20}, \lambda) \qquad l_2 = g(l_{10}, l_{20}, \lambda) \tag{4.6}$$

where l_{10} and l_{20} are the initial values (associated, e.g. with $\lambda \to \infty$). For each fixed value of λ, equation (4.6) defines a point transformation in the (l_1,l_2) plane; and in coordinates $(\Phi;\, u)$ it is a pure translation, with components $\int_\infty^\lambda \mathrm{d}\lambda/\mu$ and $\int_\infty^\lambda \lambda\,\mathrm{d}\lambda/\mu$: as a result, trajectories $(\Phi$ = constant) are transformed into new trajectories. Moreover, the formulae (4.6) turn out to be algebraic (rational in λ, linear in μ); they can be identified with Bäcklund transformation formulae, and λ with the spectral parameter (Scott *et al* 1973, Ablowitz *et al* 1973).

In paper I we considered the u-translation generator G_1, defined by

$$\delta_1 u = 1 \qquad \delta_1\Phi = 0$$

and the second generator G_2

$$\delta_2 u = 0 \qquad \delta_2\Phi = 1.$$

Clearly, the infinitesimal BT (4.5) is the linear combination

$$\frac{1}{\mu}(\lambda G_1 + G_2)$$

of generators, with coefficients dependent on λ. Solving (4.5) for $\mathrm{d}l_1$ and $\mathrm{d}l_2$ accordingly yields the system

$$\begin{aligned} \mu\frac{\mathrm{d}l_1}{\mathrm{d}\lambda} &= (\lambda\delta_1 l_1 + \delta_2 l_1) = m_1\left(\frac{\lambda - l_2}{l_1 - l_2}\right) \\ \mu\frac{\mathrm{d}l_2}{\mathrm{d}\lambda} &= (\lambda\delta_1 l_2 + \delta_2 l_2) = m_2\left(\frac{l_1 - \lambda}{l_1 - l_2}\right). \end{aligned} \tag{4.7}$$

In coordinates $S \equiv (l_1 + l_2)$, $P \equiv l_1 l_2$, this becomes

$$\begin{aligned} \mu\frac{\mathrm{d}S}{\mathrm{d}\lambda} &= \lambda S'(u) - P'(u) \\ \mu\frac{\mathrm{d}P}{\mathrm{d}\lambda} &= \lambda P' + PS' - SP' \end{aligned} \tag{4.8}$$

where $S'(u)$ and $P'(u)$ are implicit functions of S and P, which will be calculated later.

4.2. The spectral function

Although the differential system (4.8) may be solved in a systematic way (e.g. through a limited series expansion in the neighbourhood of a root of $\mu^2(\lambda)$) since its solution is rational in λ, linear in μ, the solution may be found much more simply by the consideration of the spectral function.

Let us consider a function $\chi(u)$, defined by (Weiss 1983, p 1408; 'A higher order KdV equation')

$$\frac{\chi'(u)}{\chi} \equiv M = \frac{[P'(u) - \lambda S'(u) - \mu]}{2[\lambda^2 - \lambda S + P]} \tag{4.9}$$

where λ is an arbitrary parameter (the spectral parameter). We show later that $\chi(u)$ also satisfies a Schrödinger equation, with polynomial dependence on the parameter λ.

First, it will be necessary to establish the form of the algebraic relations determining $S'(u)$ and $P'(u)$ as implicit functions of S, P

$$\begin{aligned} (SS'^2 - 2P'S') &= \frac{(m_1^2 - m_2^2)}{(l_1 - l_2)} = F(P; S) \\ (PS'^2 - P'^2) &= \frac{(m_1^2 l_2 - m_2^2 l_1)}{(l_1 - l_2)} = G(P; S) \end{aligned} \tag{4.10}$$

where the functions F, G turn out to be polynomial in S and P

$$\begin{aligned} F \equiv a_6 S[S^4 - 4PS^2 + 3P^2] + a_5[S^4 - 3PS^2 + P^2] + a_4 S[S^2 - 2P] \\ +a_3[S^2 - P] + a_2 S + a_1 \end{aligned} \tag{4.11}$$

$$\begin{aligned} G \equiv a_6 P[S^4 - 3PS^2 + P^2] + a_5 PS[S^2 - 2P] + a_4 P[S^2 - P] \\ +a_3 PS + a_2 P - a_0. \end{aligned} \tag{4.12}$$

They satisfy a second-order, linear partial differential system

$$G_S + PF_P = 0 \qquad G_P = F_S + SF_P \tag{4.13}$$

where the lower indices S, P denote partial differentiation.

The polynomial equations (4.10), with F, G defined by (4.11), (4.12) and (4.4), constitute the nonlinear differential system which was the subject of the preceding sections.

We shall also need an expression for the second derivatives $S''(u)$ and $P''(u)$: they can be found by differentiation of (4.10), and they are given by

$$2S''(u) = -F_P \qquad (2P'' - S'^2) = -G_P. \tag{4.14}$$

We can now calculate the second derivative $\chi''(u)$ of the spectral function

$$\frac{2\chi''(u)}{\chi} \equiv 2[M'(u) + M^2] = \frac{(P'' - \lambda S'')}{(\lambda^2 - \lambda S + P)} + \frac{[\mu^2 - (P' - \lambda S')^2]}{2(\lambda^2 - \lambda S + P)^2}. \tag{4.15}$$

It is remarkable that the quantity on the right-hand side is polynomial in λ: we obtain, after simplification

$$\frac{4\chi''(u)}{\chi} \equiv 4[M'(u) + M^2] = a_6[\lambda^2 + 2\lambda S + (3S^2 - 2P)] + a_5[\lambda + 2S] + a_4 \tag{4.16}$$

which has the form of a Schrödinger equation for $\chi(u)$, with quadratic dependence on the spectral parameter λ. It may be rewritten

$$\frac{4\chi''(u)}{\chi} = \text{ polynomial part of : } \mu^2(\lambda) \left\{ \frac{1}{\lambda^4} + \frac{2S}{\lambda^5} + \frac{(3S^2 - 2P)}{\lambda^6} \right\}. \tag{4.17}$$

4.3. Generalization to a partial differential equation integrable by the inverse scattering transform method

The previous results suggest considering the following partial differential system

$$\begin{aligned} &\psi_{xx} = v\psi \\ &\psi_t = A_x\psi - 2A\psi_x \end{aligned} \tag{4.18}$$

where $A \equiv (\lambda^2 - \lambda S + P)$, and λ is an arbitrary constant. The associated condition of integrability (the result of the elimination of ψ) reads

$$v_t = A_{xxx} - 4vA_x - 2Av_x. \tag{4.19}$$

When v is given by the right-hand side of equation (4.16), this equation becomes a polynomial of the first degree in λ

$$\lambda\Omega_1 + \Omega_0 = 0 \tag{4.20}$$

where

$$\begin{aligned} \Omega_1 &= a_6\left\{\frac{S_t}{2} + \partial_x[S(3P - 2S^2)]\right\} + a_5\partial_x\left(P - \frac{3}{2}S^2\right) - a_4S_x + S_{xxx} \\ \Omega_0 &= 3a_6\left\{\frac{(3SS_t - P_t)}{6} + P_x(S^2 - P) + PSS_x\right\} + a_5\left(\frac{S_t}{2} + 2SP_x + PS_x\right) \\ &\quad + a_4P_x - P_{xxx}. \end{aligned} \tag{4.21}$$

The equations

$$\Omega_0 = 0 \qquad \Omega_1 = 0 \tag{4.22}$$

constitute a partial differential system for the functions $S(x, t)$, $P(x, t)$ (which no longer involves the arbitrary constant λ). It is the condition of the integrability of ψ, defined by (4.18) which may thus be viewed as a Lax pair, showing that the system (4.22) must be integrable by the inverse scattering transform method. The ordinary differential equations (4.10) correspond, of course, to the stationary solutions of (4.22), and $\psi = \chi\, \mathrm{e}^{\mu t}$.

We expect that the Bäcklund transformation of the ordinary differential system must have a simple form in terms of the spectral function χ; for instance, in the problem considered by Weiss (1983), where $a_6 = 0$, the BT results from the inversion of the spectral function χ, i.e. the change of sign of M, without changing the value of the spectral parameter λ; denoting $\hat{\chi}$, $\hat{M}$, etc the transformed quantities, the BT is determined by the fundamental relations

$$\hat{\chi} = \frac{1}{\chi} \qquad \hat{M} = -M \qquad \hat{\lambda} = \lambda \qquad \hat{\mu} = -\mu. \tag{4.23}$$

Note added in proof. In general (i.e. when $a_6 \neq 0$), there are two sets of limiting values of (S, P) as $\lambda \to \infty$:(S_0, P_0) and (S_1, P_1), depending on the sign of μ, and the solution of the differential system (4;8) involves *both* corresponding spectral functions χ_0, χ_1 (or, rather, their logarithmic derivatives M_0, M_1). Denoting for conciseness $M(\lambda, S, P)$ the function defined by (4;9), and $M_i = M(\lambda, S_i, P_i)$ $(i = 0, 1)$, the solution of the differential system (4;8) is

$$\begin{cases} 2S = 2(M_0 - M_1) + (S_0 + S_1) \\ 2P = -4M_0M_1 + 2\left[S_0 + S_1 + \dfrac{a_5}{2}\right](M_0 - M_1) + [\lambda^2 + \lambda(S_0 + S_1 + a_5) + C_0] \end{cases}$$

where $C_0 = (S_0^2 + S_0S_1 + S_1^2) + a_5(S_0 + S_1) + a_4$, and we have chosen $a_6 = 1$. S_0, P_0 may be viewed as the two integration constants, and S_1, P_1 are constants which are symmetrically and algebraically related to S_0, P_0.

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